

TROPICAL CURVES AND INTEGRABLE PIECEWISE LINEAR MAPS

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ABSTRACT. We present applications of tropical geometry to some integrable piecewise-linear maps, based on the lecture given by one of the authors (R. I.) at the workshop “Tropical Geometry and Integrable Systems” (University of Glasgow, July 2011), and on some new results obtained afterward. After a brief review on tropical curve theory, we study the spectral curves and the isolevel sets of the tropical periodic Toda lattice and the periodic Box-ball system.

1. INTRODUCTION

1.1. Background — integrable systems and algebraic geometry. Let us show an example of the remarkable application of complex algebraic geometry to integrable systems.

Fix $N \in \mathbb{Z}_{>1}$, and let \mathbb{Z}_N be the quotient ring $\mathbb{Z}/N\mathbb{Z}$. The N -periodic Toda lattice equation is a famous integrable system given by

$$a'_n = b_n - b_{n-1}, \quad b'_n = b_n(a_{n+1} - a_n), \quad (1.1)$$

on the *phase space* $M = \{(a_n, b_n)_{n \in \mathbb{Z}_N}\} \simeq \mathbb{C}^{2N}$. Here we write a'_n for a derivation of $a_n = a_n(t)$ by the time t .¹ To solve this equation, we use an important property that there are $N+1$ algebraically independent polynomial functions h_j ($j = 1, \dots, N+1$) on M , which are conserved by the equation. Fix $c = (c_1, \dots, c_{N+1}) \in \mathbb{C}^{N+1}$ and define the subset of M by

$$M_c = \{m \in M \mid h_j(m) = c_j \ (j = 1, \dots, N+1)\}.$$

This is called the *isolevel set* invariant under the time evolution. By the definition, M_c is an algebraic variety. Since the holomorphic function on M_c corresponds to the solution, we want to know what kind of algebraic variety it is. Let γ_c be the algebraic curve given by

$$y^2 + y(x^N + c_1 x^{N-1} + \dots + x c_{N-1} + c_N) + c_{N+1} = 0. \quad (1.2)$$

The curve γ_c is called the *spectral curve*, which is also invariant under the time evolution. When γ_c is smooth (i.e. γ_c is a hyperelliptic curve of genus $N-1$), we have the followings [1, 10]:

- (i) The isolevel set M_c is isomorphic to an affine part of the Jacobian $\text{Jac}(\gamma_c)$ of γ_c .
 - (ii) The solution is written in terms of the corresponding Riemann’s theta function.
- Moreover, the flow of the equation is linearized on $\text{Jac}(\gamma_c)$.

Key words and phrases. Tropical geometry, spectral curve, isolevel set, Jacobian, Toda lattice, Box-ball system.

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¹The original form of the Toda lattice equation is $x''_n = e^{x_{n+1}-x_n} - e^{x_n-x_{n-1}}$. One obtains the above form via the transformation: $a_n = x'_n$, $b_n = e^{x_{n+1}-x_n}$.

Let us explain more detail in the case of $N = 2$. With $c = (c_1, c_2, c_3) \in \mathbb{C}^3$, we fix the three conserved functions as

$$h_1 = a_1 + a_2 = c_1, \quad h_2 = a_1 a_2 - b_1 - b_2 = c_2, \quad h_3 = b_1 b_2 = c_3.$$

By erasing a_2 and b_1 in these relations, we obtain $b_2^2 + b_2(a_1^2 - a_1 c_1 + c_2) + c_3 = 0$. This is nothing but the defining equation of γ_c via $(x, y) = (-a_1, b_2)$. The map $\phi : M_c \rightarrow \text{Jac}(\gamma_c)$ is a composition of

$$\begin{array}{ccccc} M_c & \rightarrow & \gamma_c & \xrightarrow{\text{AJ}} & \text{Jac}(\gamma_c) \\ (a_1, a_2, b_1, b_2) & \mapsto & (-a_1, b_2) := P & \mapsto & \int_{P_0}^P \omega \end{array},$$

where AJ is the Abel-Jacobi map with a base point $P_0 \in \gamma_c$ and the holomorphic one form ω on γ_c . The image of ϕ is $\text{Jac}(\gamma_c)$ minus two points corresponding to the infinity points of γ_c .

As one sees in this example, the key to solve the equation is to find the spectral curve whose Jacobian is related to the isolevel set.

1.2. Tropicalization. Let K be an algebraic closed field with a valuation $\text{val} : K \setminus \{0\} \rightarrow \mathbb{R}$ as

$$\text{val}(ab) = \text{val}(a) + \text{val}(b), \quad \text{val}(a + b) \geq \min[\text{val}(a), \text{val}(b)],$$

for $a, b \in K$. The *tropicalization* is the map from K to \mathbb{R} by this valuation. For a polynomial $f = \sum_{m \in I} c_m x^m \in K[x_1, \dots, x_n]$ where I is a finite subset of $(\mathbb{Z}_{\geq 0})^n$, we define its *tropicalization* as

$$F = \min_{m \in I} [\text{val}(c_m) + m \cdot X],$$

where we set $m = (m_1, \dots, m_n)$, $x^m = \prod_i x_i^{m_i}$ and $m \cdot X = \sum_i m_i X_i$. We call F as a *tropical polynomial* in n variables X_1, \dots, X_n . In this manner, min-plus algebra is obtained as the tropicalization of K . Tropical geometry is algebraic geometry of min-plus algebra [9], which can be interpreted as the tropicalization of the algebraic geometry on K .

On the other hand, among integrable systems, there are some rational maps (on K) whose tropicalization gives non-trivial interesting piecewise linear maps (on \mathbb{R}). We are interested in the case of $K = \overline{\mathbb{C}((t))}$, the field of Puiseux series in t , and we expect a similar story as §1.1, i.e. integrable structure of the piecewise-linear maps is described by tropical geometry.

1.3. Contents. We are interested in two piecewise-linear systems, the tropical periodic Toda lattice (trop-pToda), and the periodic Box-boll system (pBBS). These systems are obtained from known integrable rational maps: the former is the tropicalization of discrete Toda lattice as the name suggests, and the latter is the special case of the tropical KdV equation. It is natural to study them with tropical geometry, since the integrability of the original rational maps is described by complex algebraic geometry as the previous example. But we would like to emphasize that it is highly non-trivial problem, because the tropicalization is a very formal limiting procedure and nothing is ensured about how the original complex structure becomes.

This article is organized as follows: we introduce the basic notions of tropical curve theory in §2, including tropical Jacobians for tropical curves, and tropical theta functions, by following [12]. In §3, we review the general solution of the trop-pToda based on [6, 7, 8]. The general isolevel set is isomorphic to the tropical Jacobian, and the corresponding solution is written in terms of the tropical theta function. In §4, we present new results on the evolution equation and the spectral curve of the pBBS. The initial value problem of the pBBS was already solved by applying crystal theory [14] or combinatorics [16], where

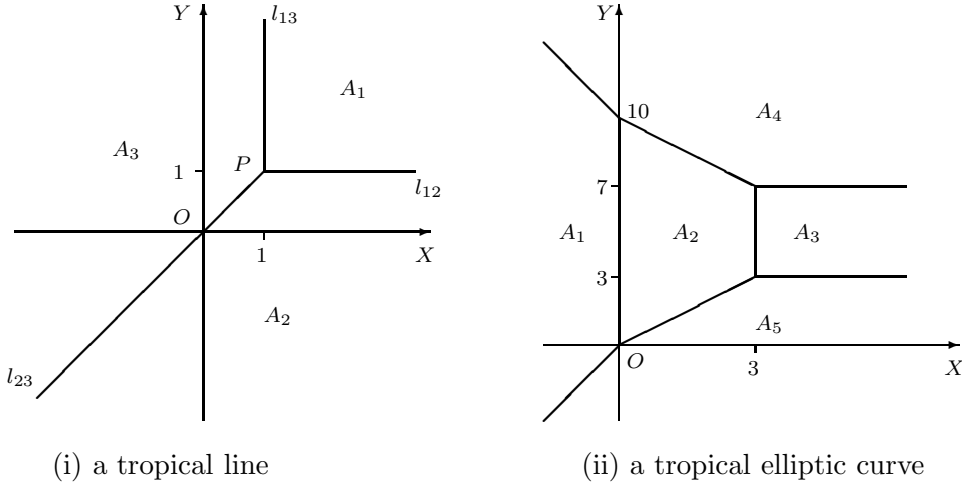


FIGURE 1. Tropical curves

the time evolution is linearized on the high-dimensional real torus, and the tropical theta functions appear in the solution. We explore the tropical geometrical aspect of the pBBS via the tropical KdV equation. We explicitly give the tropical spectral curve of the pBBS, and show that the above real torus is really the Jacobian of the tropical curve.

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2. TROPICAL CURVE THEORY

2.1. Tropical curves. In this article we consider affine tropical curves in \mathbb{R}^2 given by tropical polynomials of two variables as

$$F(X, Y) = \min_{i \in I} [C_i + n_i X + m_i Y] \quad C_i \in \mathbb{R}, n_i, m_i \in \mathbb{Z}_{\geq 0},$$

where I is a finite set. The tropical curve Γ given by a tropical polynomial $F(X, Y)$ is defined by

$$\Gamma = \{(X, Y) \in \mathbb{R}^2 \mid F(X, Y) \text{ is } \textit{indifferentiable}\}.$$

The meaning of “ (X, Y) is indifferentiable” is that $F(X, Y)$ is accomplished by more than one terms in $F(X, Y)$ at (X, Y) .

Example 2.1. See Figure 1 for examples of tropical curves. (i) is given by $F(X, Y) = \min(X, Y, 1)$, and (ii) is given by $F(X, Y) = \min[2Y, Y + 2X, Y + X, Y + 3, 10]$. Let us explain the notion of “indifferentiable” in (i). Let A_1 , A_2 and A_3 be three open domains divided by the three boundaries l_{12} , l_{23} and l_{13} , and let P be the intersection point $l_{12} \cap l_{23} \cap l_{13}$. The function $F(X, Y) = \min(X, Y, 1)$ is “differentiable” at $(X, Y) \in A_1 \cup A_2 \cup A_3$, since we have $F(X, Y) = 1$ in A_1 , $F(X, Y) = Y$ in A_2 and $F(X, Y) = X$ in A_3 . On the other hand, $F(X, Y)$ is “indifferentiable” at $(X, Y) \in l_{12} \cup l_{23} \cup l_{13}$, since at least two of X , Y and 1 become the minimum. For instance, $F(X, Y) = Y = 1$ on $l_{12} \setminus \{P\}$, and $F(X, Y) = X = Y = 1$ at P .

The edges in tropical curves have rational slopes, and we associate each vertex with a *primitive tangent vector* which is a tangent vector given by a pair of coprime integers. The

primitive tangent vector is uniquely determined up to sign. (If one of the integers is zero, then let another be ± 1 .)

Definition 2.2. The tropical curve $\Gamma \subset \mathbb{R}^2$ is smooth if the following two conditions hold:

- (a) all vertices in Γ are 3-valent.
- (b) For each 3-valent vertex v , let ξ_1, ξ_2, ξ_3 be the primitive tangent vectors which are outgoing from v . Then these vectors satisfy $\xi_1 + \xi_2 + \xi_3 = (0, 0)$ and $|\xi_i \wedge \xi_j| = 1$ for $i, j \in \{1, 2, 3\}$, $i \neq j$. (For two vectors $\xi = (n_1, n_2)$ and $\xi' = (n'_1, n'_2)$, we set $\xi \wedge \xi' = n_1 n'_2 - n_2 n'_1$.)

When a tropical curve Γ is smooth, the *genus* of Γ is $\dim H_1(\Gamma, \mathbb{Z})$.

The two tropical curves at Figure 1 are smooth, and the genera are zero and one respectively.

A smooth tropical curve is equipped with the *metric structure* as follows (We omit the metric structure for non-smooth tropical curves for simplicity. See [12].):

Definition 2.3. Assume Γ is a smooth tropical curve. Let $E(\Gamma)$ be the set of edges in Γ , and let ξ_e be the primitive tangent vector of $e \in E(\Gamma)$. We define the *length* of edges $l : E(\Gamma) \rightarrow \mathbb{R}_{\geq 0}$ by

$$e \mapsto l(e) = \frac{\|e\|}{\|\xi_e\|},$$

where $\|\cdot\|$ is any norm in \mathbb{R}^2 .

2.2. Abelian integral and tropical Jacobian. Let Γ be a smooth tropical curve whose genus g is not zero. We fix g generators B_1, \dots, B_g of the fundamental group of Γ . We define the *bilinear form* of two paths p and p' on Γ by

$$\langle p, p' \rangle = \text{“the oriented length of the common part of } p \text{ and } p' \text{”}.$$

Here “oriented” means the sign depending on the directions of the two paths on the common part.

Example 2.4. See Figure 2 for the smooth tropical curve Γ given by

$$F(X, Y) = \min(2Y, Y + 3X, Y + 2X, Y + X + 1, Y + 4, 11).$$

The genus of Γ is 2, and we fix the basis B_1 and B_2 of the fundamental group of Γ as depicted. The bilinear forms of B_1 and B_2 take the values as

$$\langle B_1, B_1 \rangle = 20, \quad \langle B_1, B_2 \rangle = -7, \quad \langle B_2, B_2 \rangle = 14.$$

Let us demonstrate how to compute $\langle B_1, B_2 \rangle$: the common part of B_1 and B_2 is the edge PQ , and we have $l(PQ) = 7$. Moreover, the directions of B_1 and B_2 on PQ are opposite, and we obtain $-l(PQ) = -7$ as the oriented length of $B_1 \cap B_2$.

Now we introduce the *abelian integral* and the *tropical Jacobian* of Γ :

Definition 2.5. Fix $P_0 \in \Gamma$. The *abelian integral* $\psi : \Gamma \rightarrow \mathbb{R}^g$ is given by

$$P \mapsto \psi(P) = (\langle B_i, \widehat{P_0 P} \rangle)_{i=1, \dots, g},$$

where $\widehat{P_0 P}$ is a path from P_0 to P . The map ψ induces the map from a set of divisors $\text{Div}(\Gamma)$ on Γ to \mathbb{R}^g ;

$$\sum_{i \in I} n_i P_i \mapsto \sum_{i \in I} n_i \psi(P_i),$$

where I is a finite set and $n_i \in \mathbb{Z}$.

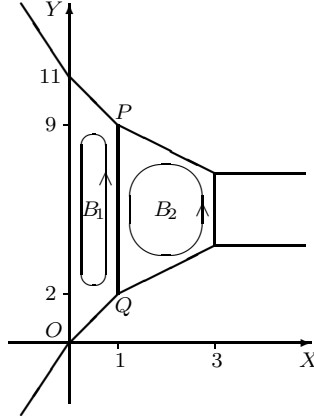


FIGURE 2. Tropical curve of genus 2

Definition 2.6. The period matrix Ω of Γ is given by

$$\Omega = (\langle B_i, B_j \rangle)_{i,j=1,\dots,g} \in \text{Mat}(g; \mathbb{R}). \quad (2.1)$$

The tropical Jacobian variety $J(\Gamma)$ of Γ is the g -dimensional real torus given by

$$J(\Gamma) = \mathbb{R}^g / \Omega \mathbb{Z}^g. \quad (2.2)$$

Example 2.7. The tropical curve of genus 1 depicted at Figure 1 (ii) has the period matrix $\Omega = 20$, and the Jacobian is $\mathbb{R} / 20\mathbb{Z}$. As for the tropical curve of genus 2 depicted at Figure 2, the period matrix and the Jacobian are as

$$\Omega = \begin{pmatrix} 20 & -7 \\ -7 & 14 \end{pmatrix}, \quad J(\Gamma) = \mathbb{R}^2 / \Omega \mathbb{Z}^2.$$

Remark 2.8. The matrix Ω is symmetric and positive definite by definition, and $J(\Gamma)$ is a tropical analogue of Jacobian variety. By removing all infinite edges of Γ , we obtain the maximal compact subgraph Γ^{cpt} of Γ . The map ψ is not injective since $\widehat{P_0}P$ is not unique, but the induced map $\Gamma^{\text{cpt}} \rightarrow J(\Gamma)$ becomes injective. When $g = 1$, ψ induces $\Gamma^{\text{cpt}} \xrightarrow{\sim} J(\Gamma)$.

Remark 2.9. There is a well-defined notion of rational equivalence class in $\text{Div}(\Gamma)$. Let $\text{Pic}^k(\Gamma)$ be the rational equivalence class of $\text{Div}^k(\Gamma)$, where $\text{Div}^k(\Gamma) \subset \text{Div}(\Gamma)$ is a set of divisors of degree k . Then we have a commutative diagram:

$$\begin{array}{ccc} \text{Div}^k(\Gamma) & \rightarrow & J(\Gamma) \\ & \searrow & \uparrow \beta \\ & & \text{Pic}^k(\Gamma) \end{array}$$

where the map β is an isomorphism [12].

2.3. Tropical theta function. Fix a positive integer g and a symmetric and positive definite matrix $\Omega \in \text{Mat}(g; \mathbb{R})$. (Here the matrix Ω is not always a period matrix of some tropical curve.)

Definition 2.10. The *tropical theta function* $\Theta(\mathbf{Z}; \Omega)$ of $\mathbf{Z} \in \mathbb{R}^g$ is defined by

$$\Theta(\mathbf{Z}; \Omega) = \min_{\mathbf{n} \in \mathbb{Z}^g} \left\{ \mathbf{n} \cdot \left(\frac{1}{2} \Omega \mathbf{n} + \mathbf{Z} \right) \right\}.$$

We call the g -dimensional real torus given by

$$J_\Omega = \mathbb{R}^g / \Omega \mathbb{Z}^g \quad (2.3)$$

the *principally polarized tropical abelian variety*. (If Ω is the period matrix of a tropical curve Γ , then J_Ω is nothing but the tropical Jacobian $J(\Gamma)$.)

It is easy to see the following:

Lemma 2.11. *The function $\Theta(\mathbf{Z}) = \Theta(\mathbf{Z}; \Omega)$ satisfies the quasi-periodicity:*

$$\Theta(\mathbf{Z} + \Omega \mathbf{m}) = -\mathbf{m} \cdot \left(\frac{1}{2} \Omega \mathbf{m} + \mathbf{Z} \right) + \Theta(\mathbf{Z}) \quad \mathbf{m} \in \mathbb{Z}^g. \quad (2.4)$$

Remark 2.12. Recall the Riemann's theta function:

$$\theta(\mathbf{z}; W) = \sum_{\mathbf{n} \in \mathbb{Z}^g} \exp(\pi \sqrt{-1} \mathbf{n} \cdot (W \mathbf{n} + 2\mathbf{z})) \quad \mathbf{z} \in \mathbb{C}^g, \quad (2.5)$$

where $W \in \text{Mat}(g; \mathbb{C})$ is symmetric and $\text{Im} W$ is positive definite. This function satisfies the periodicity and quasi-periodicity:

$$\begin{aligned} \theta(\mathbf{z} + \mathbf{m}; W) &= \theta(\mathbf{z}; W), \\ \theta(\mathbf{z} + K \mathbf{m}; W) &= \exp(-\pi \sqrt{-1} \mathbf{m} \cdot (W \mathbf{m} + 2\mathbf{z})) \theta(\mathbf{z}; W), \end{aligned} \quad (2.6)$$

for $\mathbf{m} \in \mathbb{Z}^g$. Remark that only the quasi-periodicity remains in the tropical case.

3. TROPICAL PERIODIC TODA LATTICE

3.1. Introduction. The tropical periodic Toda lattice (trop-pToda) is given by the piecewise-linear evolution equation:

$$\begin{aligned} Q_j^{t+1} &= \min(W_j^t, Q_j^t - X_j^t), & X_j^t &= \min_{0 \leq k \leq N-1} \left(\sum_{l=1}^k (W_{j-l}^t - Q_{j-l}^t) \right), \\ W_j^{t+1} &= Q_{j+1}^t + W_j^t - Q_j^{t+1} \end{aligned} \quad (3.1)$$

on the phase space

$$\mathcal{T} = \{(Q_j, W_j)_{j \in \mathbb{Z}_N} \mid \sum_{j=1}^N Q_j < \sum_{j=1}^N W_j\} \subset \mathbb{R}^{2N}.$$

(In [6], this system is called the ultradiscrete periodic Toda lattice, where “ultradiscrete” means “tropical” in our present terminology.)

Remark 3.1. The trop-pToda is the tropicalization of the discrete N -periodic Toda lattice [4] given by

$$q_j^{t+1} = q_j^t + w_j^t - w_{j-1}^{t+1}, \quad w_j^{t+1} = \frac{q_{j+1}^t w_j^t}{q_j^{t+1}}$$

on the phase space $\{(q_j, w_j)_{j \in \mathbb{Z}_N}\} \simeq K^{2N}$ under the setting:

$$\sum_{j=1}^N (\text{val}(w_j^t) - \text{val}(q_j^t)) > 0, \quad Q_j^t = \text{val}(q_j^t), \quad W_j^t = \text{val}(w_j^t).$$

See [11, Prop. 2.1] for the detail, where the strategy is essentially same as Lemma 4.6.

The Toda lattice equation (1.1) is a continuous limit $\delta \rightarrow 0$ of the above discrete Toda lattice, with $w_j^t = \delta^2 b_j$ and $q_j^t = 1 + \delta a_j'$. Here δ is a unit of the discrete time.

The system (3.1) has $N + 1$ conserved tropical polynomials H_k ($k = 1, \dots, N + 1$) on \mathcal{T} . Here are some of them:

$$\begin{aligned} H_1 &= \min_{1 \leq j \leq N} (Q_j, W_j), \\ H_2 &= \min\left(\min_{1 \leq i < j \leq N} (Q_i + Q_j), \min_{1 \leq i < j \leq N} (W_i + W_j), \min_{1 \leq i, j \leq N, j \neq i, i-1} (Q_i + W_j)\right), \\ H_N &= \min\left(\sum_{j=1}^N Q_j, \sum_{j=1}^N W_j\right), \\ H_{N+1} &= \sum_{j=1}^N (Q_j + W_j). \end{aligned} \tag{3.2}$$

3.2. General solution. Fix $C = (C_k)_{k=1, \dots, N+1} \in \mathbb{R}^{N+1}$, and define the isolevel set \mathcal{T}_C by

$$\mathcal{T}_C = \{\tau \in \mathcal{T} \mid H_k(\tau) = C_k \text{ } (k = 1, \dots, N + 1)\}. \tag{3.3}$$

We are to describe the general solution to (3.1) and the isolevel set \mathcal{T}_C in terms of tropical geometry.

Let Γ_C be the tropical curve given by tropical polynomial

$$F(X, Y) = \min(2Y, Y + \min(NX, (N - 1)X + C_1, \dots, X + C_{N-1}, C_N), C_{N+1}). \tag{3.4}$$

We call Γ_C the spectral curve of the trop-pToda. Remark that $F(X, Y)$ corresponds to the tropicalization of the l.h.s of (1.2) (the defining equation for the algebraic curve γ_c .)

We set L , λ_k and p_k for $k = 0, \dots, N - 1$ as

$$\begin{aligned} L &= C_{N+1} - 2(N - 1)C_1, \\ \lambda_0 &= 0, \quad \lambda_k = C_{k+1} - C_k \quad k = 1, \dots, N - 1, \\ p_0 &= L, \quad p_k = L - 2 \sum_{j=1}^{N-1} \min(\lambda_k, \lambda_j) \quad k = 1, \dots, N - 1. \end{aligned} \tag{3.5}$$

The curve Γ_C is smooth if and only if $\lambda_1 < \lambda_2 < \dots < \lambda_{N-1}$ and $p_k > 0$ for $k = 1, \dots, N - 1$. Assume Γ_C is smooth, then the genus g is $g = N - 1$. See Figure 3 for Γ_C , where we set $C_1 = 0$ for simplicity.

Fix the basis B_1, \dots, B_g of the fundamental group $\pi_1(\Gamma_C)$ as Figure 3. The period matrix Ω (2.1) of Γ_C is obtained as

$$\Omega_{ij} = \begin{cases} p_{i-1} + p_i + 2(\lambda_i - \lambda_{i-1}) & i = j \\ -p_i & j = i + 1 \\ -p_j & i = j + 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

and we get the tropical Jacobian of Γ_C as

$$J(\Gamma_C) = \mathbb{R}^g / \Omega \mathbb{Z}^g. \tag{3.7}$$

Theorem 3.2. *When Γ_C is smooth, we have the following:*

(i) [7, Th. 3.5] Fix $\mathbf{Z}_0 \in \mathbb{R}^g$ and define $T_n^t = \Theta(\mathbf{Z}_0 + \boldsymbol{\lambda}t - L\mathbf{e}_1n; \Omega)$, where

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2 - \lambda_1, \dots, \lambda_g - \lambda_{g-1}), \quad \mathbf{e}_1 = (1, 0, \dots, 0) \in \mathbb{R}^g.$$

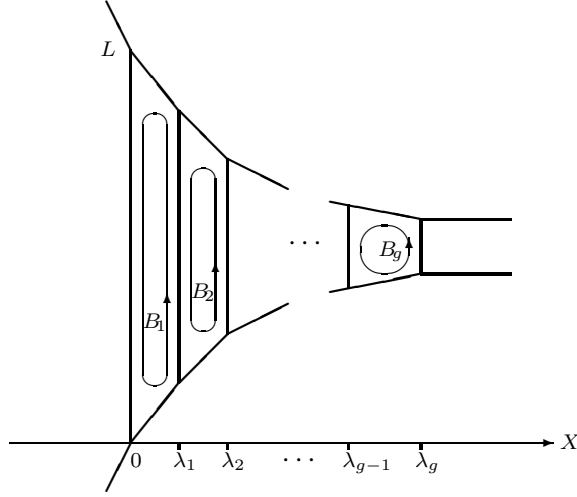


FIGURE 3. Spectral curve for the trop-pToda

The solution for the trop-pToda is given by

$$\begin{aligned} Q_n^t &= T_{n-1}^t + T_n^{t+1} - T_{n-1}^{t+1} - T_n^t + C_1, \\ W_n^t &= T_{n-1}^{t+1} + T_{n+1}^t - T_n^t - T_n^{t+1} + L + C_1. \end{aligned} \quad (3.8)$$

(ii) [8, Th. 1.3] This solution induces the isomorphism $J(\Gamma_C) \xrightarrow{\sim} \mathcal{T}_C$. In particular, the time evolution of the trop-pToda is linearized on $J(\Gamma_C)$, whose velocity is λ .

One of the keys to prove Theorem 3.2 (i) is the following lemma:

Lemma 3.3. [7, Prop. 2.10] The function T_n^t satisfies

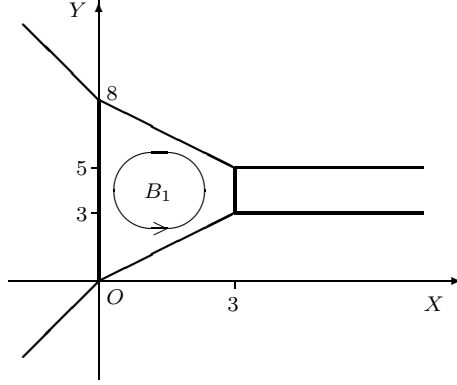
$$T_n^{t+1} + T_n^{t-1} = \min[2T_n^t, T_{n-1}^{t+1} + T_{n+1}^{t-1} + L].$$

This identity corresponds to a tropicalization of the bilinear form. This lemma is proved by applying the tropical version of Fay's trisecant identity for tropical theta functions. See [7] for the detail.

Example 3.4. The case of $N = 2$. The curve Γ_C is smooth if and only if $C_3 > 2C_2 > 4C_1$. In this simplest case, we can explicitly construct the isomorphism α :

$$\begin{array}{ccccc} \mathcal{T}_C & \xrightarrow{\alpha} & \Gamma_C^{\text{cpt}} & \xrightarrow{\psi} & J(\Gamma_C) \\ (Q_1, W_1, Q_2, W_2) & \mapsto & P = (\min(Q_2, W_1), Q_1 + W_1) & \mapsto & \langle B_1, \widehat{P_0 P} \rangle. \end{array}$$

The solution (3.8) induces the inverse map of $\psi \circ \alpha$. Let us consider the case of $C = (0, 3, 8)$, where Γ_C is depicted as



The following is an example of linearization, where one sees $\lambda = (3)$. We set $P_0 = O$:

	$\mathcal{T}_C = \{(Q_1, W_1, Q_2, W_2)\}$	$\xrightarrow{\alpha} \Gamma_C^{\text{cpt}}$	$\xrightarrow{\psi} J(\Gamma_C) \simeq \mathbb{R}/16\mathbb{Z}$
$t=0$	$(3, 4, 0, 1)$	$(0, 7)$	9
$t=1$	$(3, 1, 0, 4)$	$(0, 4)$	12
$t=2$	$(1, 0, 2, 5)$	$(0, 1)$	15
$t=3$	$(0, 2, 3, 3)$	$(2, 2)$	$2 \equiv 18$
$t=4$	$(0, 5, 3, 0)$	$(3, 5)$	$5 \equiv 21$

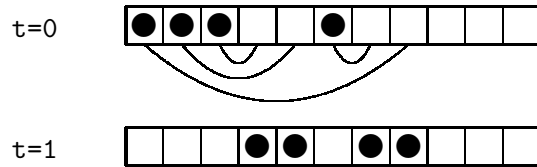
For general $N > 2$, the isomorphism $\mathcal{T}_C \xrightarrow{\sim} J(\Gamma_C)$ is regarded as a composition of the injective map $\alpha : \mathcal{T}_C \rightarrow \text{Div}_{\text{eff}}^g(\Gamma_C)$ and the abelian integral ψ , but α becomes too complicated.

4. PERIODIC BBS

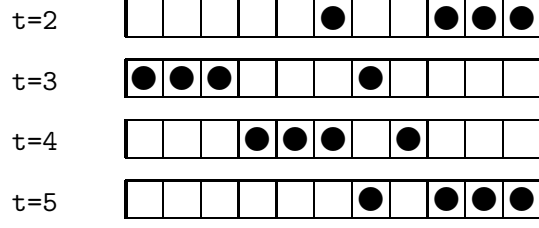
4.1. Introduction. The periodic Box-ball system (pBBS) is a cellular automaton defined by adding a periodic boundary condition [18] to the original (infinite) Box-ball system [17]. Let L be the number of boxes aligned on an oriented circle. Put $M < L/2$ balls into the boxes, assuming that each box can accommodate one ball at most. Move the balls with the following rule which defines the time evolution from t to $t + 1$:

- (1) Connect “an occupied box whose immediate right is empty” and the empty box with an arc. Do the same for all such boxes.
- (2) In the rest, do the same as (1) by ignoring the connected boxes and arcs.
- (3) Continue the same procedure as (2) until all occupied boxes are connected with empty boxes.
- (4) Move all balls to the connected empty boxes.

In the process, the term “right” is understood along the direction of the orientation of the circle. This evolution is determined uniquely and invertible. Let us show an example of $L = 11$ and $M = 4$ in the following, where we identify the left and right boundaries with thick lines. The above rule works as



and we obtain more as follows:

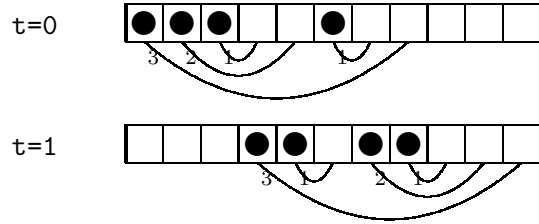


One can observe that the larger series of balls overtakes the smaller one repeatedly. We call a series of balls as a *soliton*. When a state has g solitons, we call the state as a g -soliton state. The system has a finite configuration space with $\binom{L}{M}$ states, thus any state comes back to itself in a finite time. We identify an occupied box and an empty box with 1 and 0 respectively, and define the phase space of the pBBS by

$$\mathcal{U}_{\text{BBS}} = \left\{ (U_k)_{k \in \mathbb{Z}_L} \mid U_k \in \{0, 1\}, \sum_{k=1}^L U_k < \frac{L}{2} \right\}. \quad (4.1)$$

We write U_k^t for the number of balls in the k -th box at time t .

The pBBS has the conserved quantities $\mu = (\mu_1, \mu_2, \mu_3, \dots)$ constructed as the by-product of the evolution rule; let μ_1 (resp. μ_2) be the number of the arcs drawn at the step (1) (resp. (2)), and so on. Let us calculate μ of the above example, where the arc with the number i contributes to μ_i :



We see that $\mu = (2, 1, 1)$ is invariant under the evolution.

By the definition, μ_1 is the number of the solitons in a state. We introduce an equivalent expression λ to μ by

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\mu_1}), \quad \lambda_j = \#\{\mu_k \mid \mu_k \geq \mu_1 + 1 - j\}.$$

(Do not confuse these λ_j with those at (3.5).) We have $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{\mu_1}$ and $\sum_i \mu_i = \sum_{j=1}^{\mu_1} \lambda_j < \frac{L}{2}$. In the above example, we get $\lambda = (1, 3)$. We regard λ as the map from g -soliton states in \mathcal{U}_{BBS} to $(\mathbb{Z}_{\geq 0})^g$, and write $\lambda(U)$ for the image of a g -soliton state $U \in \mathcal{U}_{\text{BBS}}$.

4.2. General solution. We fix $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_g)$ as $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_g$, and define the isolevel set:

$$\mathcal{U}_{\text{BBS}, \lambda} = \{U \in \mathcal{U}_{\text{BBS}} \mid \lambda(U) = \lambda\}.$$

The symmetry of the pBBS is described by the $\widehat{\mathfrak{sl}}_2$ crystal, and the general solution was obtained as follows (For an introductory review, see [5].):

Theorem 4.1. [13, 14] For $\lambda = (\lambda_1 < \dots < \lambda_g)$, define $p_k \in \mathbb{Z}_{>0}$ ($k = 1, \dots, g$) and $A = (A_{ij})_{i,j=1,\dots,g} \in \text{Mat}(g; \mathbb{Z})$ by

$$p_k = L - 2 \sum_{j=1}^g \min(\lambda_k, \lambda_j), \quad A_{ij} = p_i \delta_{ij} + 2 \min[\lambda_i, \lambda_j]. \quad (4.2)$$

Then A is symmetric and positive definite. Define the g -dimensional torus $J(L, \lambda)$ as

$$J(L, \lambda) = \mathbb{R}^g / A \mathbb{Z}^g. \quad (4.3)$$

Then we have the following:

(i) There is one-to-one correspondence between the isolevel set $\mathcal{U}_{\text{BBS}, \lambda}$ and the integer points in $J(L, \lambda)$ induced by what is called the Kerov-Kirillov-Reshetekhin bijection. Let Φ be the corresponding embedding $\Phi : \mathcal{U}_{\text{BBS}, \lambda} \rightarrow J(L, \lambda)$. Via Φ , the time evolution of the pBBS is linearized on $J(L, \lambda)$, whose velocity is λ .

(ii) For $U^0 = (U_k^0)_{k \in \mathbb{Z}_L} \in \mathcal{U}_{\text{BBS}, \lambda}$, set $\mathbf{Z}_0 = \Phi(U^0) - \frac{\mathbf{p}}{2}$. Then $U^t = (U_k^t)_{k \in \mathbb{Z}_L}$ is written in terms of tropical theta function $\Theta(\mathbf{Z}) = \Theta(\mathbf{Z}; A)$ as

$$\begin{aligned} U_k^t = & -\Theta(\mathbf{Z}_0 - k\mathbf{v}_1 + t\lambda) + \Theta(\mathbf{Z}_0 - (k-1)\mathbf{v}_1 + t\lambda) \\ & + \Theta(\mathbf{Z}_0 - k\mathbf{v}_1 + (t+1)\lambda) - \Theta(\mathbf{Z}_0 - (k-1)\mathbf{v}_1 + (t+1)\lambda), \end{aligned}$$

where $\mathbf{v}_1 = (1, 1, \dots, 1) \in \mathbb{R}^g$.

Remark 4.2. The g -dimensional torus (4.3) is the principally polarized tropical abelian variety (Definition 2.10).

Remark 4.3. There is a family of commutative and invertible time evolutions $\{T_m\}_{m \in \mathbb{Z}_{>0}}$ on $\mathcal{U}_{\text{BBS}, \lambda}$ [2, Th. 3.2], and T_m induces the linear motion on $J(L, \lambda)$ of the velocity $\mathbf{v}_m = (\min[m, \lambda_j])_{j=1, \dots, g} \in \mathbb{Z}^g$ [14, Th. 3.11]. For $m \geq \lambda_g$, T_m gives the original evolution of the pBBS, namely, $\mathbf{v}_m = \lambda$.

We will use the following lemma in §4.4:

Lemma 4.4. Fix $\lambda = (\lambda_1 < \dots < \lambda_g)$.

(i) There is a state $U_0 \in \mathcal{U}_{\text{BBS}, \lambda}$ without soliton scattering, i.e. the set composed of the lengths of g solitons coincides with $\{\lambda_1, \dots, \lambda_g\}$.

(ii) For any state $U \in \mathcal{U}_{\text{BBS}, \lambda}$, there is a sequence of evolutions $T := T_{l_1}^{n_1} T_{l_2}^{n_2} \dots T_{l_k}^{n_k}$ ($1 \leq l_1, \dots, l_k \leq \lambda_g$, $n_1, \dots, n_k \in \mathbb{Z} \setminus \{0\}$) such that $T(U) = U_0$.

Proof. (i) It follows from the condition $\sum_k U_k < \frac{L}{2}$ of \mathcal{U}_{BBS} . (ii) Due to Theorem 4.1 and Remark 4.3, it is enough to prove that $\oplus_{1 \leq m \leq \lambda_g} \mathbb{Z} \mathbf{v}_m$ includes the basis $\{\mathbf{e}_i = (\underbrace{0, \dots, 0}_{i-1}, 1, \underbrace{0, \dots, 0}_{g-i}) \mid i = 1, \dots, g\}$ of \mathbb{Z}^g . We actually have

$$\sum_{k=i}^g \mathbf{e}_k = -\mathbf{v}_{\lambda_{i-1}} + \mathbf{v}_{\lambda_{i-1}+1} \quad i = 1, \dots, g,$$

where we assume $\lambda_0 = 0$ and $\mathbf{v}_0 = (0, \dots, 0) \in \mathbb{Z}^g$. Hence the claim follows. \square

In the following sections, we are to clarify the tropical geometrical origin of $J(L, \lambda)$.

4.3. Tropical periodic KdV equation and pBBS. The discrete KdV equation is given by the evolution equation [3]:

$$u_k^{t+1} + \frac{\delta}{u_{k-1}^{t+1}} = u_{k-1}^t + \frac{\delta}{u_k^t} \quad (k, t \in \mathbb{Z}), \quad (4.4)$$

where δ is a constant element. We assume the periodic boundary condition $u_{k+L}^t \equiv u_k^t$ for each k, t . Naturally, we regard the index k as an element of \mathbb{Z}_L .

The discrete KdV equation (4.4) is equivalent to the matrix equation:

$$R^{t+1} S^{t+1} = S^t R^t, \quad (4.5)$$

where $S^t = S^t(y)$ and $R^t = R^t(y)$ are elements of $\text{Mat}(L; K[y])$ which are defined by

$$S^t = \begin{pmatrix} u_1^t & 1 & & & \\ & u_2^t & 1 & & \\ & & \ddots & \ddots & \\ & & & u_{L-1}^t & 1 \\ (-1)^{L-1}y & & & & u_L^t \end{pmatrix}, \quad R^t = \begin{pmatrix} \frac{\delta}{u_1^t} & 1 & & & \\ & \frac{\delta}{u_2^t} & 1 & & \\ & & \ddots & \ddots & \\ & & & \frac{\delta}{u_{L-1}^t} & 1 \\ (-1)^{L-1}y & & & & \frac{\delta}{u_L^t} \end{pmatrix}.$$

Put $X^t(y) := R^t(y)S^t(y)$. Then the equation (4.5) is rewritten as

$$X^{t+1}S^t = S^tX^t, \quad (4.6)$$

which implies that the characteristic polynomial $f(x, y) = \det(X^t(y) + x \cdot \text{id.})$ is invariant under the time evolution. ($-x$ is the eigenvalue of $X(y)$). The algebraic curve defined by the polynomial f is called the *spectral curve* of periodic discrete KdV. The following lemma follows from [15, Th. II.1].

Lemma 4.5. *Let M be the maximum integer satisfying $M < \frac{L}{2}$. The polynomial f is written as:*

$$f(x, y) = y^2 + y(c_M x^M + \cdots + c_1 x + c_0) + (x + \delta)^L,$$

where c_i ($i = 0, 1, \dots, M$) are rational functions in $\mathbb{R}_{>0}(u_n^t, \delta)$ or $\mathbb{R}_{<0}(u_n^t, \delta)$.

The pBBS is obtained as a tropical KdV equation with the periodic boundary condition. Let $u_k^t, \delta \in K = \overline{\mathbb{C}((t))}$ and $\text{val}(u_k^t) = U_k^t$, $\text{val}(\delta) = 1$. Let \mathcal{U} be the subset of \mathbb{Q}^L defined by

$$\mathcal{U} = \{(U_k)_{k \in \mathbb{Z}_L} \mid U_k \in \mathbb{Q}, \sum_{k=1}^L U_k < \frac{L}{2}\},$$

which includes \mathcal{U}_{BBS} .

Lemma 4.6. *Let*

$$P_{k,m} := \prod_{l=0}^m \left(\frac{\delta}{u_{k-l}^t u_{k-l-1}^t} \right), \quad P := P_{1,L} = \prod_{l=1}^L \left(\frac{\delta}{u_l^t u_{l-1}^t} \right).$$

When $(U_k^t)_{k \in \mathbb{Z}_L}$ and $(U_k^{t+1})_{k \in \mathbb{Z}_L}$ are contained in \mathcal{U} , the periodic discrete KdV (4.4) is equivalent to:

$$u_k^{t+1} = \frac{\delta}{u_k^t} \left(1 + \frac{1 - P}{P_{k,0} + P_{k,1} + \cdots + P_{k,L-1}} \right). \quad (4.7)$$

Proof. Using (4.4) recursively, we have

$$\begin{aligned} u_k^{t+1} &= u_{k-1}^t + \frac{\delta}{u_k^t} - \frac{\delta}{u_{k-1}^{t+1}} = u_{k-1}^t + \frac{\delta}{u_k^t} - \frac{\delta}{u_{k-2}^t + \frac{\delta}{u_{k-1}^t} - \frac{\delta}{u_{k-2}^{t+1}}} \\ &= \cdots = u_{k-1}^t + \frac{\delta}{u_k^t} - \frac{\delta}{u_{k-2}^t + \frac{\delta}{u_{k-1}^t} - \frac{\delta}{\ddots - \frac{\delta}{u_{k-L}^{t+1}}}}. \end{aligned}$$

We can regard this continued fraction as a quadratic equation in u_k^{t+1} because of the periodic boundary condition $u_{k-L}^{t+1} = u_k^{t+1}$. The two solutions of this quadratic equation are expressed as:

$$u_k^{t+1} = \frac{\delta}{u_k^t}, \quad \text{or} \quad u_k^{t+1} = \frac{\delta}{u_k^t} \left(1 + \frac{1 - P}{P_{k,0} + P_{k,1} + \cdots + P_{k,L-1}} \right).$$

However, the first solution contradicts the condition $(U_k^t)_{k \in \mathbb{Z}_L}, (U_k^{t+1})_{k \in \mathbb{Z}_L} \in \mathcal{U}$. In fact, if $u_k^{t+1} = \delta/u_k^t$, the following equation should be true:

$$\sum_k U_k^{t+1} = \text{val} \left(\prod_k u_k^{t+1} \right) = \text{val} \left(\prod_k \frac{\delta}{u_k^t} \right) = L - \sum_k U_k^t,$$

which implies $\sum_k U_k^t \geq L/2$ or $\sum_k U_k^{t+1} \geq L/2$. \square

Let $K_{>0} \subset K$ be the semifield defined by

$$K_{>0} := \{c_{-n/d} t^{-n/d} + c_{(-n+1)/d} t^{-(n+1)/d} + \dots \mid c_{-n/d} > 0, n/d \in \mathbb{Q}_{>0}\}.$$

For $a, b \in K_{>0}$, it follows that $\text{val}(a+b) = \min[\text{val}(a), \text{val}(b)]$, which is not always true on K .

Assume $u_k^t \in K_{>0}$ for all n, t . By taking the valuation of (4.7), we have the following proposition:

Proposition 4.7. *The tropicalization of the periodic discrete KdV equation is given by the piecewise-linear map*

$$U_k^{t+1} = \min \left[1 - U_k^t, \max_{m=0,1,\dots,L-1} \left[\sum_{j=1}^{m+1} U_{k-j}^t - \sum_{j=1}^m (1 - U_{k-j}^t) \right] \right] \quad (4.8)$$

on \mathcal{U} . We refer to this system as the tropical KdV equation. This evolution equation is closed on the phase space \mathcal{U}_{BBS} .

Proof. If $u_k^t \in K_{>0}$ and $(U_k^t)_k \in \mathcal{U}$, we have

$$\text{val}(P_{k,0} + P_{k,1} + \dots + P_{k,L-1}) = \min[\text{val}(P_{k,0}), \text{val}(P_{k,1}), \dots, \text{val}(P_{k,L-1})]$$

and $\text{val}(1-P) = 0$. Then, direct calculations conclude (4.8) immediately. Next, we prove that $U_k^t \in \{0, 1\}$ implies $U_k^{t+1} \in \{0, 1\}$. For this, it is enough to prove

$$N_k^t := \max_{m=0,1,\dots,L-1} \left[\sum_{j=1}^{m+1} U_{k-j}^t - \sum_{j=1}^m (1 - U_{k-j}^t) \right] \geq 0.$$

It is easily checked that N_k^t equals to the number of arcs (§4.1) which straddle the boundary between the $(k-1)$ -th box and the k -th box at time t . Especially, this number is non-negative. \square

Corollary 4.8. *Let*

$$U_k^t := \{\text{the number of ball in the } n\text{-th box at time } t\} \in \{0, 1\}.$$

Then, the evolution equation (4.8) is equivalent to the pBBS.

Proof. By the definition of pBBS introduced in §4.1, we find

$$\begin{aligned} U_k^{t+1} = 1 &\iff U_k^t = 0 \text{ and } N_k^t > 0, \\ U_k^{t+1} = 0 &\iff U_k^t = 1 \text{ or } N_k^t = 0, \end{aligned}$$

which is equivalent to $U_{k+1}^{t+1} = \min[1 - U_k^t, N_k^t]$. \square

We obtain the following as a corollary of Lemma 4.5.

Corollary 4.9. *Let M be the maximum integer satisfying $M < \frac{L}{2}$. The tropical spectral curve Γ of the tropical KdV equation is given by*

$$F(X, Y) = \min[2Y, Y + \min_{j=0,\dots,M} [jX + C_j], XL, L]. \quad (4.9)$$

Here C_j ($j = 1, \dots, M$) are tropical functions on \mathcal{U} .

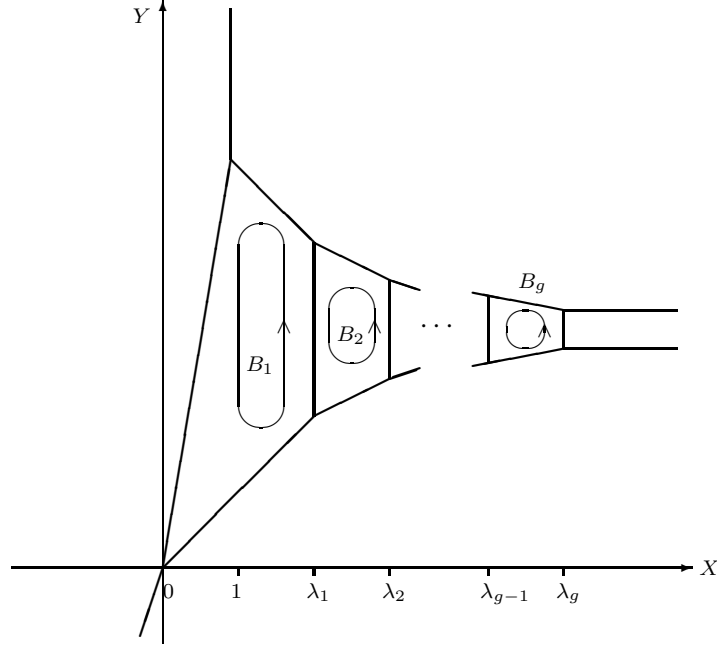


FIGURE 4. Spectral curve for pBBS

4.4. Tropical spectral curve of pBBS. The main result of this subsection is the following.

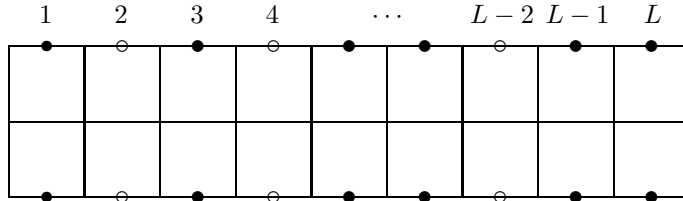
Proposition 4.10. *Fix the conserved quantity of the pBBS as $\lambda = (\lambda_1 < \dots < \lambda_g)$. Then the corresponding spectral curve Γ of the piecewise-linear map (4.8) on $\mathcal{U}_{\text{BBS},\lambda}$ is given by*

$$F(X, Y) = \min[2Y, Y + \min[gX, (g-1)X + C_{g-1}, \dots, X + C_1, C_0], XL, L]. \quad (4.10)$$

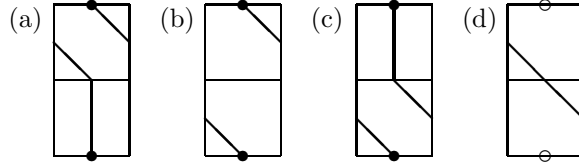
where $C_j = \sum_{i=1}^{g-j} \lambda_i$ for $j = 0, \dots, g-1$.

See Figure 4 for the tropical curve Γ .

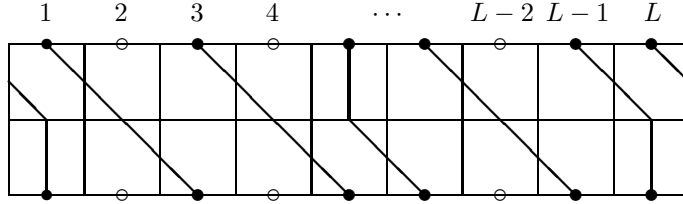
To prove Proposition 4.10, we have to describe the detail of C_j by applying [15]. Define $\mathcal{L} = \{1, \dots, L\}$ and $\mathcal{J}_j = \{I \subset \mathcal{L} \mid |I| = j, \text{ no consecutive numbers in } I\}$. Consider a $2 \times L$ lattice and fix $I \in \mathcal{J}_j$. On the lattice we set \circ (resp. \bullet) at the top and bottom of the k -th column for $k \in I$ (resp. $k \in \mathcal{L} \setminus I$). Here is the example of $L = 9$ and $I = \{2, 4, 7\} \in \mathcal{J}_3$:



We identify the two vertical boundaries, and tile this lattice with the following 4 patterns:



in such a way that (a)-(c) are located at $k \in \mathcal{L} \setminus I$ and (d) is at $k \in I$ to have just $L - j$ non-intersecting paths each of which starts from upper \bullet and ends at lower \bullet . Note that $\mathcal{J}_j = \emptyset$ for $j > M$, and that no tilling exists when I includes consecutive numbers. The following is one of the possible tillings of the above example:



We write F_I for such a tilling, and call F_I a *possible tilling* for I . Define functions on \mathcal{U} by

$$\xi(U_k; F_I) = \begin{cases} 1 - U_k & \text{if } F_I \text{ has (c) at } k\text{-th column} \\ U_k & \text{if } F_I \text{ has (a) at } k\text{-th column} \\ 0 & \text{otherwise,} \end{cases} \quad k \in \mathcal{L}. \quad (4.11)$$

For instance, the above tilling F_I gives $\xi(U_1; F_I) = U_1$, $\xi(U_5; F_I) = 1 - U_5$, $\xi(U_9; F_I) = U_9$ and $\xi(U_k; F_I) = 0$ for other k . We remark $\xi(U_k; F_I) \in \{0, 1\}$ on $\mathcal{U}_{\text{BBS}} \subset \mathcal{U}$.

As a tropicalization of [15, Th. II.1], we obtain the following:

Lemma 4.11. *The conserved quantities C_j are written as*

$$C_j = \min_{I \in \mathcal{J}_j} \min_{F_I} \left[\sum_{k \in \mathcal{L}} \xi(U_k; F_I) \right] \quad j = 0, \dots, M. \quad (4.12)$$

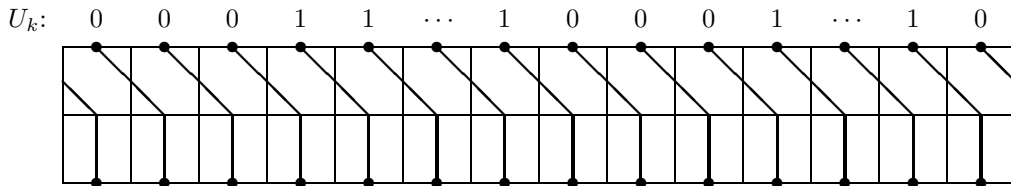
Here is a key lemma for Proposition 4.10:

Lemma 4.12. *Fix $\lambda = (\lambda_1 < \lambda_2 < \dots < \lambda_g)$. On $\mathcal{U}_{\text{BBS}, \lambda} \subset \mathcal{U}_{\text{BBS}}$, C_j (4.12) are written as follows:*

(i) $C_j = \sum_{i=1}^{g-j} \lambda_i$ for $j = 0, \dots, g-1$, and $C_g = 0$. (ii) $C_j = 0$ for $g < j \leq M$.

Proof. We are to show the tilling F_I such that $C_j = \sum_{k \in \mathcal{L}} \xi(U_k; F_I)$ explicitly. The point is to find a tilling F_I which realizes $\xi(U_k; F_I) = 0$ for as many $k \in \mathcal{L}$ as possible.

(i) When $j = 0$, \mathcal{J}_0 is empty and the $2 \times N$ lattice does not have \circ . Thus a possible tilling is given by filling all lattices with (a) or with (c). Since $|\lambda| < \frac{L}{2}$, the tilling with (a) as

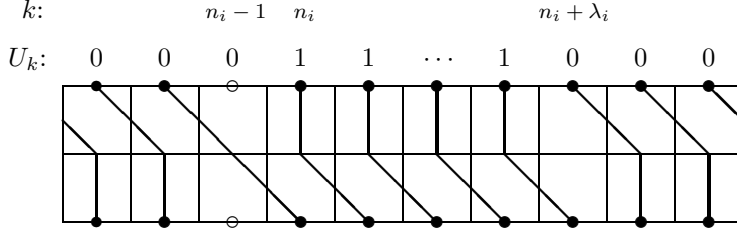


gives the minimum and $C_0 = |\lambda|$ is realized. We write F_0 for the above tiling.

Let us show the cases of $j = 1, \dots, g$. Due to Remark 4.3 and Lemma 4.4, it is enough to consider the state without soliton scattering, since C_j 's are conserved by the evolutions T_m . Set

$$n_i = \text{“the coordinate of the soliton of length } \lambda_i \text{”} \in \mathbb{Z}/L\mathbb{Z}, \quad (4.13)$$

and define $I_j = \{n_i - 1 \ (i = g - j + 1, \dots, g)\} \in \mathcal{J}_j$. Define a tiling $F_j := F_{I_j}$ by replacing the tiles of F_0 at $n_i - 1 \leq k \leq n_i + \lambda_i$ ($n_i - 1 \in I_j$) with the tiles (b)-(d) as follows:



Then we have

$$\xi(U_k; F_j) = \begin{cases} 1 & n_i \leq k \leq n_i + \lambda_i - 1; n_i - 1 \in I_j \\ 0 & \text{otherwise,} \end{cases}$$

and obtain $\sum_{k \in \mathcal{L}} \xi(U_k; F_j) = \sum_{j=1}^{g-k} \lambda_j$. By the induction on j , it is easy to show that this is the minimum, and the claim follows.

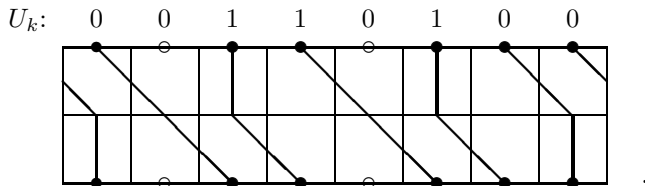
(ii) It is enough to show that there is $I \in \mathcal{J}_j$ and F_I such that $C_j = \sum_{k \in \mathcal{L}} \xi(U_k; F_I) = 0$ for the state $(U_k) \in \mathcal{U}_{\text{BBS}, \lambda}$ for $j = g + 1, \dots, M$. From $j = g + 1$ to M , such I and F_I are recursively constructed as follows.

If there is $m_a \in I_{j-1}$ as $I = I_{j-1} \cup \{m_a - 2\} \in \mathcal{J}_j$, define $I_j = I$. Otherwise, there certainly exists a subset $J = \{m_a - 3, m_a + 2n + 3, m_a + 2l \ (l = 0, \dots, n)\}$ of I_{j-1} for some $m_a \in \mathcal{L}$ and n as $0 \leq n < M$. Then define $I_j = I_{j-1} \cup \{m_a - 3, m_a + 2n + 3, m_a + 2l + 1 \ (l = -1, 0, \dots, n)\} \setminus J \in \mathcal{J}_j$.

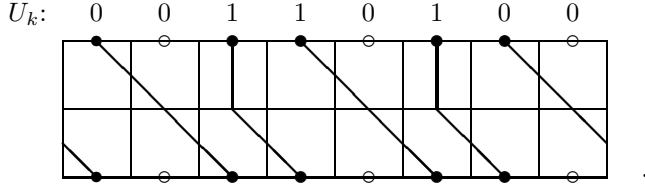
In the first case of I_j , define a possible tiling F_j by replacing the tiles of F_{j-1} at $k = m_a - 1$ and $k = m_a - 2$ with (b) and (d) respectively (if needed, replace the tile at $k = m_a - 3$ with (b)). Since these replacements do not change the quantity of the function ξ (4.11), $\xi(U_k; F_g) = \xi(U_k; F_{g+1}) = 0$ for $k = m_a - 1, m_a - 2$ ($m_a - 3$), $C_j = 0$ follows from $C_{j-1} = 0$.

In the second case of I_j , define a possible tiling F_j by replacing the tiles of F_{j-1} at $k \in \{m_a + 2l + 1 \ (l = -1, 0, \dots, n)\}$ with (d) and the tiles at $k \in \{m_a + 2l \ (l = -1, 0, \dots, n + 1)\}$ with (b). These replacements do not change the quantity of the function ξ , and $C_j = 0$ follows. (See the following example.) \square

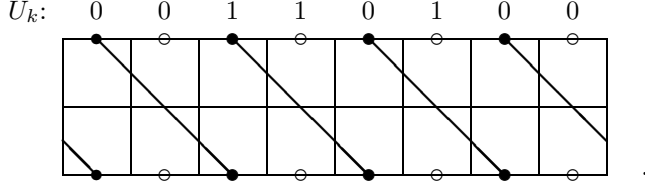
Example 4.13. The case of $L = 8$, $\lambda = (1, 2)$ and $(n_1, n_2) = (3, 6)$. Here is the tiling F_2 with $I_2 = \{2, 5\} \in \mathcal{J}_2$, which gives $C_2 = 0$:



By setting $I_3 = \{2, 5, 8\} \in \mathcal{J}_3$ ($m_a = 2$), we can define F_3 which gives $C_3 = 0$ as



There is no $m_a \in I_3$ such that $I_3 \cup \{m_a - 2\} \in \mathcal{I}_4$. Thus we set $I_4 = \{2, 4, 6, 8\}$ ($m_a = 5$, $n = 0$) and define F_4 as



Proof. (Proposition 4.10) Fix $\lambda = (\lambda_1 < \dots < \lambda_g)$. From Lemma 4.12, the tropical polynomial (4.9) is written as

$$F(X, Y) = \min[2Y, Y + \min[\min_{j=0, \dots, g-1} [jX + C_j], \min_{j=g, \dots, M} [jX]], XL, L],$$

and the corresponding tropical curve Γ has three infinite domains determined by $F(X, Y) = XL$, L and $2Y$, which fill the domain $D = \{(X, Y) \in \mathbb{R}^2 \mid X < 0 \text{ or } Y < 0\}$. In the rest domain $\mathbb{R}^2 \setminus D$, we have $\min_{j=g, \dots, M} [jX] = gX$. Thus the defining equation of Γ can be reduced to (4.10). \square

Though Γ is not smooth, we can calculate its period matrix Ω in the same way as Definition 2.6:

$$\Omega = (\langle \tilde{B}_k, \tilde{B}_j \rangle)_{k, j=1, \dots, g}. \quad (4.14)$$

Here we set $\tilde{B}_k = \sum_{j=1}^{g+1-k} B_j$ by using the basis B_j 's of $\pi_1(\Gamma')$ as Figure 4. Then we obtain our final result:

Proposition 4.14. *The period matrix Ω (4.14) coincides with the period matrix A (4.2). In particular, $J(L, \lambda)$ (4.3) is nothing but the tropical Jacobian $J(\Gamma)$ of Γ .*

Remark 4.15. The trop-pToda and the pBBS are closely related dynamical systems [6]. Here we just note the relation between the two tropical Jacobians studied in §3 and §4. By setting $N = g + 1$, $C_1 = 0$ and $C_{N+1} = L$ in §3, and identifying λ_i 's in the both sections, $J(L, \lambda)$ (4.3) turns out to be isomorphic to the quotient space of $J(\Gamma_C)$ (3.7) by the action $\nu : J(\Gamma_C) \rightarrow J(\Gamma_C)$; $[z] \mapsto [z + L\mathbf{e}_1]$.

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